The Emperor's New Truncation

A Modern Fairy Tale

Once upon a Time ...

Many years ago there lived an Emperor who was so exceedingly fond of fine new clothes that he spent vast sums of money on dress. To him clothes meant more than anything else in the world. He took no interest in his army, nor did he care to go to the theatre, or to drive about in his state coach, unles it was to display his new clothes. He had different robes for every singl hour of the day.

We all know how the story ends. The emperor, bare-ass naked, parades himself in front of his subjects. Only the voice of a little child declares that the emperor has no clothes.

And so it is now, with BDD. Everyone is so fond of new ideas, and they hope for new breakthroughs in PSA quantificatio that some embrace every proposed innovation with "oooooos" and "ahhhhs".

We, however, are like the child who asks himself, "But does he have anything on?

We all like fairy tales. But in this time of global need for renewable energy, it i up to each of us to insure proper and correct methods for nuclear PSA, not invisible cloth.

So forgive us if this modern fairy tale w will now tell causes us to dress in less than splendor, but assuredly covering ou private parts.

being too pedantic, the notions of truth tables, BDD, minterms, and minimal cut sets. For those with a more academic bent, please refer to the seminal paper, "Mathematical Foundations of Minimal Cut Sets" [Rauzy2000].

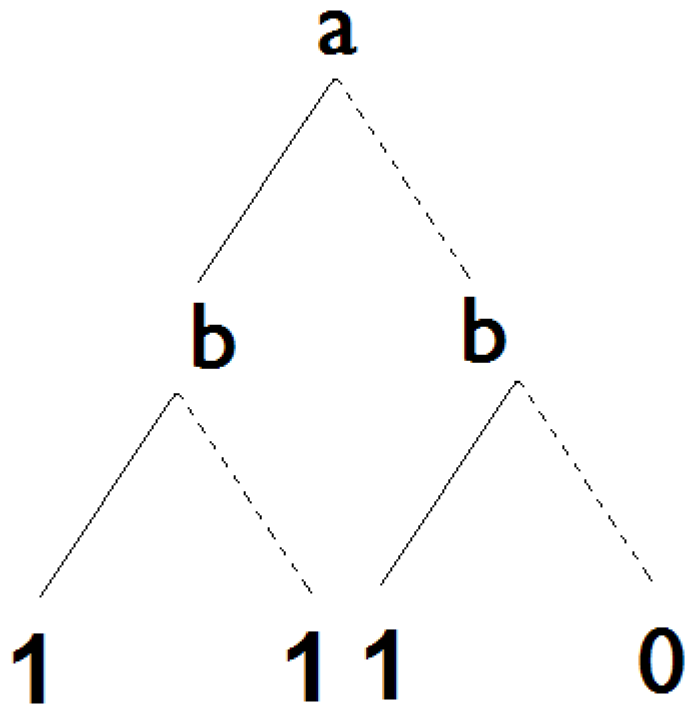
A truth table is a tabular way of representing a Boolean function. If we have a Boolean function F(a,b) = a V b, where "V" means OR, the truth table for the function would be:

a	b	a V b
True	True	True
True	False	True
False	True	True
False	False	False

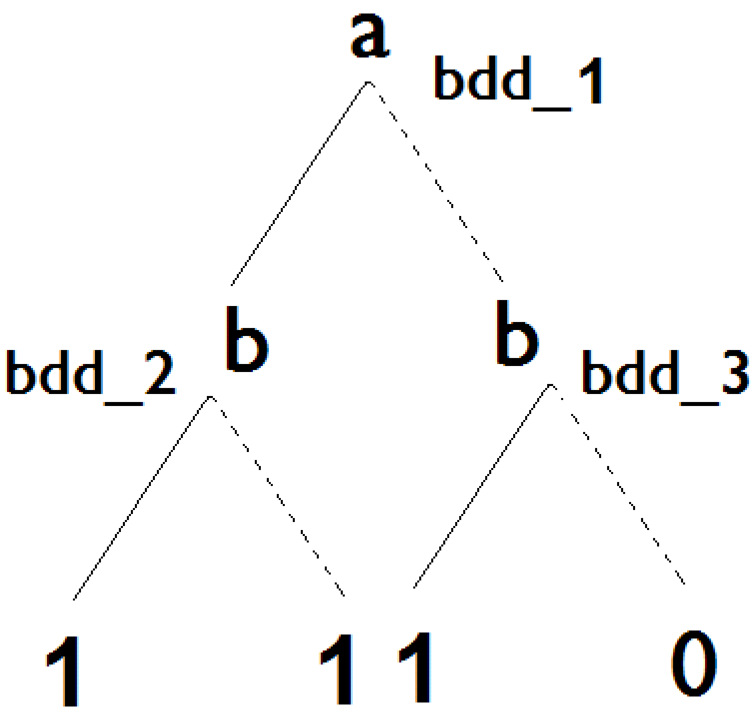
Each row of the truth table is mutually exclusive, or disjoint.

called a directed acyclic graph, DAG) of a truth table, usually, we hope, a more compact representation, built top-down.

So given the same function, F(a,b) = a V b, we might have this BDD structure:



Think of the solid lines as assigning "true" to the variable from which they emanate, the dotted lines as assigning "false", and the 1's and 0's indicating if the function is satisfied or not by the assignments. Each path in the graph is disjoint, like the rows of a truth table. We can read the graph as indicating that if a and b are true, then the function is satisfied, if a is true, but b graph with a function called "if-then-else", usually written ite(*if*, *then*, *else*), which means if a is true proceed down the left branch, if false, proceed down the right branch. So given the graph from the preceding page:



Building the BDD top-down, we can represent it with this series of functions:

BDD = bdd_1
bdd_1 = ite(a,bdd_2,bdd_3)
bdd_2 = ite(b,1,1)
bdd_3 = ite(b,1,0)

Boolean function as an equation of variables "and"ed and then "or"ed together, the same as the disjoint paths of a BDD, or the rows of a truth table.

Using variable juxtaposition as "and", "+" as "or", and "~" as "not, we can create a minterm representation of a Boolean function, F(a,b,c) ab + ~ac. The set of minterms for F are MIN(F(a,b,c)) = abc + ab~c + ~abc + ~a~bc.

These representations are logically equivalent and if you build the truth table, you will see that minterms are the "true" rows of a truth table.

a	b	c	ab	~ac	F
True	True	True	True	False	True
True	True	False	True	False	True
True	False	True	False	False	False
True	False	False	False	False	False
False	True	True	False	True	True
False	True	False	False	False	False
False	False	True	False	True	True
False	False	False	False	False	False

that truth tables, BDDs, and minterms are mathematically equivalent representations of Boolean functions. Something cannot be true about one representation that is not true about the others.

Perhaps one representation can be computed faster than another, or understood easier than another, or a given property proven more simply than another, but we are up against a basic fact about mathematics and equivalent representations: if I can say it with a truth table, then I can say it about a BDD, and if I can't, then I can't.

MCS, which are minterms without negated variables.

So given the previous function, F(a,b,c) = ab + ~ac = abc + ab~c + ~abc + ~a~bc, the min cut sets of F are MCS(F) = abc + ab + bc + c = ab c, which are the minterms, dropping negation.

The minterms of the min cut sets are MIN(MCS(F)) = abc + ab~c + ~abc + ~a~bc + a~bc. Notice that there is an additional term in green for MIN(MCS(F)), a~bc, which did not exist in the minterms.

Therefore the number of minterms in the min cut sets for a function F is always greater than, or equal to, the number of minterms of F, or mathematically, Card(MIN(MCS(F))) >= Card(MIN(F)). This is called the upper approximation, or monotone hull, of a Boolean function.

The point of the forgoing discussions was to show one very important idea: the disjoint terms of minterms, the disjoint paths of a BDD, and the disjoint rows of a truth table are all the same thing, with the disjoint terms of the minterms of min cut sets providing an upper approximation.

But why is any of this of interest to PSA?

Because we use these representations to quantify fault trees, which are a graphical representation of Boolean equations. And what we are interested in from the PSA point of view is the probability of the top event of a fault tree.

equation in minterms or a BDD, it suffice to assign probabilities to each variable, substitute multiplication for "and", addition for "or", and subtract the probability for a variable from 1 for negation. Again using F:

$$F(a,b,c) = ab + \sim ac$$

If Pr(a) = Pr(b) = Pr(c) = .1, then for MIN(F) = abc + ab~c + ~abc + ~a~bc, the Pr(F) = 1e-3 + 9e-3 + 9e-3 + 8.1e-2 = 1e-1

And for the minterms of the min cut sets MIN(MCS(F)) = abc + ab~c + ~abc + ~a~b + a~bc = 1e-3 + 9e-3 + 9e-3 + 8.1e-2 + 9.e 1.09e-1, the upper bound approximation

Notice that if we apply the same trick directly to F(a,b,c) = ab + ~ac, and conver t to min cut sets, MCS(F) = ab + c, we have the rare event approximation:

If this were the whole story, then we know, with proofs in hand, that we can calcultate the top event probabilities in PSA accurately.

But ain't life grand? In the actual fault trees used in PSA it is impossible, in the most important cases, to completely construct truth tables, minterms, min cut sets, or BDDs.

So we rely on truncation. We say we are only interested in values for minterms or min cut sets or (now) BDD paths which are greater than a certain value, the truncation cutoff, C.

Now assume that you are interested only in the MCS whose probability is greater that a given cutoff C. Then you can remove all the minterms whose MCS probability is lower than C, where the MCS probability (called MCSPr) of a minterm (or of a product in general) is defined as the product of positive literal probabilities:

$$Pr(\sim abc) = (1-Pr(a)) * Pr(b) * Pr(c)$$
$$MCSPr(\sim abc) = Pr(b) * Pr(c)$$

Let us denote by F/C the restriction of F to the minterms whose MCS probability is bigger than C and let MCS(F)/C be the set of MCS of F whose probability is bigger than c. Then the following theorem holds [Rauzy2000]:

$$MCS(F/C) = MCS(F)/C$$

Assume again that Pr(a) = Pr(b) = Pr(c) = 0.1 and that C = 0.05. Then we have:

MCS(F/C(a,b,c)) = abc + ab~c + ~abc + ~a~bc (minterms whose MCS probability is lower than C are in red).

Therefore:

MCS(F/C) = MCS(~a~bc) = { c } = MCS(F)/C (= { ab, c }) ote by the way that

IN(MCS(F/C)) = MIN(c) = abc + a~bc +
abc + ~a~b~c, and in therefore:

IN(MCS(F/C)) = MIN(F) U {A.-B.C} /
A.B.-C} (where U stands for the union
nd / stands for the set difference)

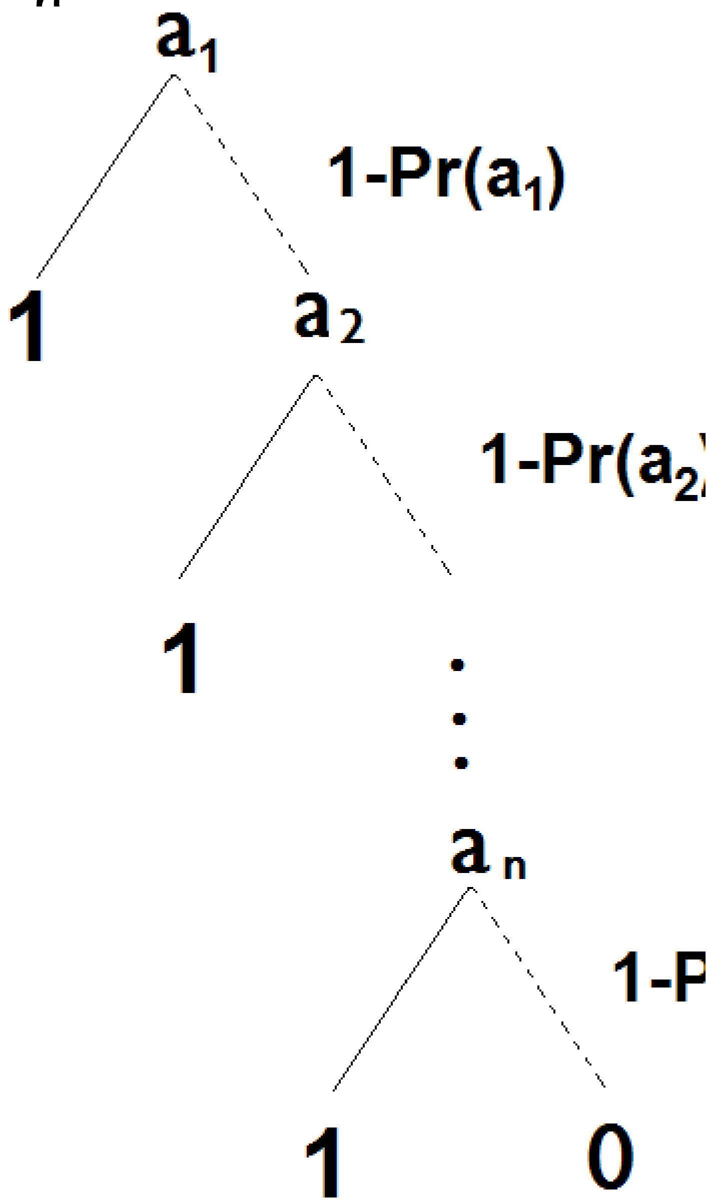
o, what you get with truncated
interms is neither an upper-
pproximation (for you remove ab~c), nor
under-approximation (for you add
~bc), but an approximation of unkown
rror.

n other words, truncating minterms,
ruth tables or BDDs works but only from
n MCS point of view, in other words, only
we include no negation.

Now The question is does probability
runcation work with BDDs?

ssume that you truncate by considering
ne probability of branches as you build t
DD. Then you go nowhere, due to the
pecific structure of BDDs. Look at the
unction F:

$$F = a_1 + a_2 + \dots a_n$$



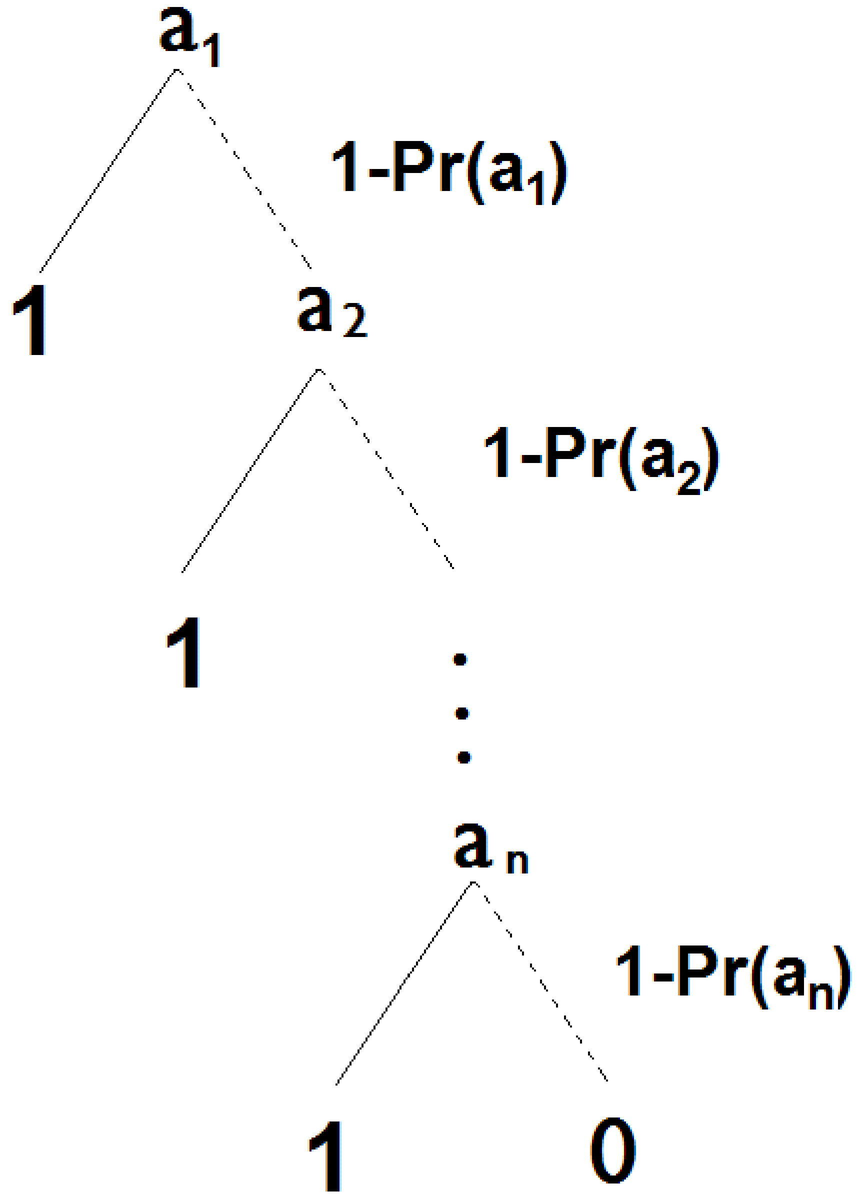
$D(F) = bdd\_1$

$d\_1 = ite(a_1, 1, bdd\_$

$d\_2 = ite(a_2, 1, bdd\_$

$d\_n = ite(a_n, 1, 0)$

y eliminating all branches whose
robability is below the cutoff C. Then as
e build this BDD,



nd we truncate counting the
probability of 0-branches, we may
reach a point where $(1-Pr(a_1))*\ldots*(1-Pr(a_i)) < C$, therefore eliminating some
perfectly valid MCS, and in doing so,
under estimate the probability of the
function F.

omplex function, we may add some
erfectly invalid MCS, for example, let H, , and G be Boolean functions such that:

$$H = \sim FG$$
$$F = a_1 + \ldots + a_n$$
$$G = a_nB$$

he truncation of F may eliminate the MC a_n}, which is part of the function G, and w nd up with an invalid MCS, a_nB, giving us ver estimation.

ogether, BDD with truncation gives us a pproximation of unknown error.

otice that this is the same result as interm truncation, which is what we wou xpect since they are equivalent epresentations. Moreover, this is the ame objection which was raised concerni he Destructive Truth Table Method and

No Free Lunch
As we said in CWTPRA, the rare event is fine
r what it does
Proposed method is supposed to help with
egation, but this is where it fails (success
anches, delete-terms, recovery actions)
No mention in other papers of the limitations
the methods
No mention of previous work
What about keeping track of what is trunacte
no mention of modules or variable ordering
Takes away from the real focus which is mod
arity and transportability
etc.

But among the crowds a little child suddenly gasped out, "But he hasn't got anything on." And the people began to whisper to one another what the child had said. "He hasn't got anything on." "There's a little child saying he hasn't got anything on." Till everyone was saying, "But he hasn't got anything on." The Emperor himself had the uncomfortable feeling that what they were whispering was only too true.